Electromagnetic field quantization in an anisotropic magnetodielectric medium with spatial–temporal dispersion

2008 J. Phys. A: Math. Theor. 41 275402

(http://iopscience.iop.org/1751-8121/41/27/275402)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **41** (2008) 275402 (12pp) doi:10.1088/1751-8113/41/27/275402

Electromagnetic field quantization in an anisotropic magnetodielectric medium with spatial–temporal dispersion

M Amooshahi and F Kheirandish

Faculty of Science, University of Isfahan, Hezar Jarib Avenue, Isfahan, Iran

E-mail: amooshahi@sci.ui.ac.ir and fardin-kh@phys.ui.ac.ir

Received 9 December 2007, in final form 22 April 2008
Published 16 June 2008
Online at stacks.iop.org/JPhysA/41/275402

Abstract

By modeling a linear, anisotropic and inhomogeneous magnetodielectric medium with two independent sets of harmonic oscillators, the electromagnetic field is quantized in such a medium. The electric and magnetic polarizations of the medium are expressed as linear combinations of the ladder operators of the harmonic oscillators modeling the magnetodielectric medium. Maxwell and the constitutive equations of the medium are obtained as the Heisenberg equations of the total system. The electric and magnetic susceptibility tensors of the medium are obtained in terms of the tensors coupling the medium with the electromagnetic field. The explicit forms of the electromagnetic field operators are obtained for a translationally invariant medium.

PACS number: 12.20.Ds

1. Introduction

One method of quantizing the electromagnetic field in the presence of an absorptive medium is known as the Green function method [1–8]. In this method by adding the noise electric and magnetic polarization densities to the classical constitutive equations of the medium, these equations are taken as definitions of electric and magnetic polarization operators. The noise polarizations are related to two independent sets of bosonic operators. Combination of Maxwell and the constitutive equations in the frequency domain gives the electromagnetic field operators in terms of the noise polarizations and the classical Green tensor. The commutation relations are imposed on the bosonic operators such that the commutation relations between electromagnetic field operators in the magnetodielectric medium become identical to those in free space. Another interesting quantization scheme of the electromagnetic field in the presence of an absorptive dielectric medium is known as the damped polarization model which is based on the Hopfield model of a dielectric [9], where the polarization of the dielectric is represented by a damped quantum field [10]. In the damped polarization model [11–13], the electric polarization of the medium is represented by a quantum field and the absorptivity character of the medium is described by the interaction between the polarization with a heat bath containing a continua of harmonic oscillators. In this method a canonical quantization is formulated for the electromagnetic field and the medium. The dielectric function of the medium is obtained in terms of the coupling function of the heat bath and the electric polarization, such that it satisfies the Kramers–Kronig relations [14]. Recently Raabe *et al* [15] have represented a unified method of quantizing the electromagnetic field in the presence of an arbitrary linear medium based on a general nonlocal conductivity tensor and using a single set of appropriate bosonic operators. This formalism recovers and generalizes the previous quantization schemes for diverse classes of linear media. In particular, the quantization of the electromagnetic field in the presence of a magnetodielectric medium is a limiting case for a weakly spatially dispersive medium in this scheme. In the present work, we generalize our approach [16, 17] to quantizing the electromagnetic field in an anisotropic magnetodielectric medium with a spatially and temporarily dispersive property. In this case, the electric and magnetic polarizations are dependent on the macroscopic electric and magnetic fields inside the medium in a nonlocal way with respect to both the position and time. The electric and magnetic polarization densities of the medium are defined as linear combinations of the ladder operators of the medium. The coefficients of these linear expansions are coupling tensors which couple the medium with the electromagnetic field. The electric and magnetic susceptibility tensors of the medium are obtained in terms of the coupling tensors. By using a Hamiltonian in which the electric and magnetic polarizations minimally couple to the displacement and magnetic fields respectively, both the Maxwell and the constitutive equations of the medium can be obtained as the Heisenberg equations of the total system. Finally, using the Laplace and Fourier transformations, we obtain the spacetime dependence of electromagnetic field operators in terms of the annihilation and creation operators of the oscillators modeling the medium.

2. A quantization scheme

In order to quantize the electromagnetic field in the presence of an anisotropic magnetodielectric medium, we enter the medium directly in the process of quantization by modeling it with two independent fields namely E and M quantum fields [16]. The E and M fields describe polarizability and magnetizability of the medium, respectively. This means that in our approach the electric and magnetic polarization densities of the medium are defined respectively as linear combinations of the ladder operators of the E and M quantum fields. We use the Coulomb gauge in this quantization scheme and do the quantization in unbounded space and in the absence of external charges. Generalization of the quantization inside a cavity with a definite volume and with known boundary conditions or in the presence of external charges is straightforward [17].

Applying the Coulomb gauge, the quantum vector potential can be expanded in terms of plane waves as

$$\vec{A}(\vec{r},t) = \int \mathrm{d}^3\vec{k} \sum_{\lambda=1}^{2} \sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_{\vec{k}}}} \left[a_{\vec{k}\lambda}(t)\,\mathrm{e}^{\imath\vec{k}\cdot\vec{r}} + a^{\dagger}_{\vec{k}\lambda}(t)\,\mathrm{e}^{-\imath\vec{k}\cdot\vec{r}}\right]\vec{e}_{\vec{k}\lambda} \tag{1}$$

where $\omega_{\vec{k}} = c|\vec{k}|$, ε_0 is the permittivity of the vacuum and $\vec{e}_{\vec{k}\lambda}(\lambda = 1, 2)$ are unit polarization vectors which satisfy the orthogonality relations

$$\vec{e}_{\vec{k}\lambda} \cdot \vec{e}_{\vec{k}\lambda'} = \delta_{\lambda\lambda'}, \qquad \vec{e}_{\vec{k}\lambda} \cdot \vec{k} = 0. \tag{2}$$

Operators $a_{\vec{k}\lambda}(t)$ and $a^{\dagger}_{\vec{k}\lambda}(t)$ are annihilation and creation operators of the electromagnetic field and satisfy the following equal time commutation rules:

$$\left[a_{\vec{k}\lambda}(t), a^{\dagger}_{\vec{k}'\lambda'}(t)\right] = \delta(\vec{k}-\vec{k}')\delta_{\lambda\lambda'}. \tag{3}$$

Quantization in the Coulomb gauge usually needs resolution of a vector field in its transverse and longitudinal parts. Any vector field $\vec{F}(\vec{r})$ can be resolved into two components, transverse and longitudinal components which are denoted respectively by $\vec{F}^{\perp}$ and $\vec{F}^{\parallel}$ [18, 19]. The transverse part satisfies the Coulomb condition $\nabla\cdot\vec{F}^{\perp}=0$ and the longitudinal component is a conservative field $\nabla\times\vec{F}^{\parallel}=0$. In the absence of external charges the displacement field is purely transverse and can be expanded in terms of the plane waves as

$$\vec{D}(\vec{r},t) = -\imath\varepsilon_0\int \mathrm{d}^3\vec{q}\sum_{\lambda=1}^{2}\sqrt{\frac{\hbar\omega_{\vec{k}}}{2(2\pi)^3\varepsilon_0}}\left[a^{\dagger}_{\vec{k}\lambda}(t)\,\mathrm{e}^{-\imath\vec{k}\cdot\vec{r}} - a_{\vec{k}\lambda}(t)\,\mathrm{e}^{\imath\vec{k}\cdot\vec{r}}\right]\vec{e}_{\vec{k}\lambda}, \tag{4}$$

where μ_0 is the permeability of the vacuum. From (3) the commutation relations between the components of the vector potential and the displacement field clearly are

$$[A_l(\vec{r},t), -D_j(\vec{r}',t)] = \imath\hbar\delta^{\perp}_{lj}(\vec{r}-\vec{r}'), \tag{5}$$

where $\delta^{\perp}_{lj}(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\,\mathrm{e}^{\imath\vec{k}\cdot(\vec{r}-\vec{r}')}\left(\delta_{lj} - \frac{k_l k_j}{|\vec{k}|^2}\right)$ is the transverse delta function.

Now we enter the medium in the process of quantization taking its contribution in the Hamiltonian of the total system (electromagnetic field plus medium) as the sum of the Hamiltonians of the E and M quantum fields

$$\begin{aligned} H_d &= H_e + H_m,\\ H_e(t) &= \sum_{\nu=1}^{3}\int \mathrm{d}^3\vec{q}\int \mathrm{d}^3\vec{k}\hbar\omega_{\vec{k}} d^{\dagger}_{\nu}(\vec{k},\vec{q},t)d_{\nu}(\vec{k},\vec{q},t),\\ H_m(t) &= \sum_{\nu=1}^{3}\int \mathrm{d}^3\vec{q}\int \mathrm{d}^3\vec{k}\hbar\omega_{\vec{k}} b^{\dagger}_{\nu}(\vec{k},\vec{q},t)b_{\nu}(\vec{k},\vec{q},t), \end{aligned} \tag{6}$$

where H_e and H_m are the Hamiltonians of the E and M fields respectively. Quantum dynamics of a dissipative harmonic oscillator interacting with an absorptive environment can be investigated by modeling the environment by a continuum of harmonic oscillators [20–28]. In the case of quantization of the electromagnetic field in the presence of a magnetodielectric medium, the electromagnetic field is the main dissipative system and the medium plays the role of the absorptive environment. Here the electromagnetic field contains a continuous set of harmonic oscillators labeled by $\vec{k}$ and $\nu = 1, 2$. Therefore to each harmonic oscillator of the electromagnetic field a continuum of oscillators should be corresponded. In the present scheme, to each harmonic oscillator of the electromagnetic field labeled by $\vec{k}$ and ν, we have corresponded two continuous sets of harmonic oscillators defined by the ladder operators $d_{\nu}(\vec{k},\vec{q},t), d^{\dagger}_{\nu}(\vec{k},\vec{q},t)$ and $b_{\nu}(\vec{k},\vec{q},t), b^{\dagger}_{\nu}(\vec{k},\vec{q},t)$ which are to describe the electric and magnetic properties of the medium respectively and satisfy the equal time commutation relations

$$\begin{aligned} \left[d_{\nu}(\vec{k},\vec{q},t), d^{\dagger}_{\nu'}(\vec{k}',\vec{q}',t)\right] &= \delta_{\nu\nu'}\delta(\vec{k}-\vec{k}')\delta(\vec{q}-\vec{q}'),\\ \left[b_{\nu}(\vec{k},\vec{q},t), b^{\dagger}_{\nu'}(\vec{k}',\vec{q}',t)\right] &= \delta_{\nu\nu'}\delta(\vec{k}-\vec{k}')\delta(\vec{q}-\vec{q}'). \end{aligned} \tag{7}$$

Summation on $\nu = 3$ in (6) is necessary, because as we will see the polarization densities of the medium are defined in terms of the ladder operators of the E and M fields and opposite to the vector potential the polarization densities are not purely transverse. In relation (6) $\omega_{\vec{k}}$ is called the dispersion relation of the magnetodielectric medium and can be chosen simply as $\omega_{\vec{k}} = c|\vec{k}|$ [16, 17]. It is remarkable that, although the medium is anisotropic in its electric and magnetic properties, we do not need take the dispersion relation as a tensor. In fact the dispersion relation has not any physical meaning here, and the Hamiltonian of the medium as (6) is merely a mathematical formulation leading to the correct form of the equation of motion of the total system, that is Maxwell and the constitutive equations of the magnetodielectric medium. The anisotropic behavior of the medium is merely expressed in definitions of the electric and magnetic polarization densities, denoted by $\vec{P}$ and $\vec{M}$ respectively. These polarization densities are written in terms of the ladder operators of the E and M fields respectively as

$$P_i(\vec{r},t) = \sum_{\nu=1}^{3} \int \frac{\mathrm{d}^3\vec{q}}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3\vec{k} \int \mathrm{d}^3 r' f_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')[d_\nu(\vec{k},\vec{q},t)\,\mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}'} + \mathrm{h.c.}]\, v_\nu^j(\vec{q}), \tag{8}$$

$$M_i(\vec{r},t) = \iota \sum_{\nu=1}^{3} \int \frac{\mathrm{d}^3\vec{q}}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3\vec{k} \int \mathrm{d}^3 r' g_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')[b_\nu(\vec{k},\vec{q},t)\,\mathrm{e}^{\iota\vec{q}\cdot\vec{r}'} - \mathrm{h.c.}]\, s_\nu^j(\vec{q}), \tag{9}$$

where

$$\vec{v}_\nu(\vec{q}) = \vec{e}_{\vec{q}\nu}, \qquad \nu = 1,2 \tag{10}$$

$$\vec{s}_\nu(\vec{q}) = \hat{q} \times \vec{e}_{\nu\vec{q}}, \qquad \nu = 1,2 \tag{11}$$

$$\vec{v}_3(\vec{q}) = \vec{s}_3(\vec{q}) = \hat{q} = \frac{\vec{q}}{|\vec{q}|}. \tag{12}$$

In (8) and (9), the tensors $f_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')$ and $g_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')$ are real-valued coupling tensors which couple the electromagnetic field with the medium. The coupling tensors play the key role in this method and are a measure for the strength of the polarizability and magnetizability of the medium macroscopically, so that we will see that the electric and magnetic susceptibility tensors of the magnetodielectric medium are obtained in terms of these coupling tensors. Also, the explicit forms of the noise polarization densities are obtained in terms of the coupling tensors and the ladder operators of the E and M fields at $t = 0$. It can be shown that when the medium tends to a nonabsorbing one, the noise densities tend to zero and this quantization scheme reduces to the usual quantization in these media [16].

Now the total Hamiltonian, i.e., electromagnetic field plus the E and M quantum fields can be proposed as one of the following forms:

$$\tilde{H}(t) = \int \mathrm{d}^3 r \left[\frac{[\vec{D}-\vec{P}]^2}{2\varepsilon_0} + \frac{[\nabla\times\vec{A} - \mu_0\vec{M}]^2}{2\mu_0} \right] + H_e + H_m, \tag{13}$$

$$\tilde{\tilde{H}}(t) = \int \mathrm{d}^3 r \left[\frac{[\vec{D}-\vec{P}]^2}{2\varepsilon_0} + \frac{[\nabla\times\vec{A}]^2}{2\mu_0} - \nabla\times\vec{A}\cdot\vec{M} \right] + H_e + H_m, \tag{14}$$

in which the electric and magnetic polarizations interact minimally with the displacement field and the magnetic field, respectively. Using both the Hamiltonians (13) and (14) gives us correctly the Maxwell and constitutive equations of the manetodielectric medium as the Heisenberg equations of the total system. Here we use the Hamiltonian (13) since it is easier to solve the coupled Maxwell and constitutive equations especially when the medium is translation invariant in its electric and magnetic properties.

3. Heisenberg equations

3.1. Maxwell equations

The Heisenberg equations for the fields $\vec{A}$ and $\vec{D}$ are

$$\frac{\partial \vec{A}(\vec{r},t)}{\partial t} = \frac{\imath}{\hbar}[\tilde{H}, \vec{A}(\vec{r},t)] = -\frac{\vec{D}(\vec{r},t) - \vec{P}^{\perp}(\vec{r},t)}{\varepsilon_0}, \tag{15}$$

$$\frac{\partial \vec{D}(\vec{r},t)}{\partial t} = \frac{\imath}{\hbar}[\tilde{H}, \vec{D}(\vec{r},t)] = \frac{\nabla \times \nabla \times \vec{A}(\vec{r},t)}{\mu_0} - \nabla \times \vec{M}(\vec{r},t), \tag{16}$$

where $\vec{P}^{\perp}$ is the transverse component of $\vec{P}$. If we define the transverse electric field $\vec{E}^{\perp}$, magnetic induction $\vec{B}$ and magnetic field $\vec{H}$ as

$$\vec{E}^{\perp} = -\frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A}, \qquad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}. \tag{17}$$

The Heisenberg equations (15) and (16) can be rewritten as

$$\vec{D} = \varepsilon_0 \vec{E}^{\perp} + \vec{P}^{\perp}, \tag{18}$$

$$\frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H}, \tag{19}$$

which are respectively the definition of the displacement field and the macroscopic Maxwell equation in the absence of external charges. In the Coulomb gauge we can take the longitudinal component of the electric field as $\vec{E}^{\parallel} = -\frac{\vec{P}^{\parallel}}{\varepsilon_0}$. According to the definitions (17) we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{20}$$

3.2. Constitutive equations of the magnetodielectric medium

Using commutation relations (7) we easily find the Heisenberg equations for operators $d_\nu(\vec{k},\vec{q},t)$ and $b_\nu(\vec{k},\vec{q},t)$ as

$$\dot{d}_\nu(\vec{k},\vec{q},t) = \frac{\imath}{\hbar}[\tilde{H}, d_\nu(\vec{k},\vec{q},t)] = -\imath\omega_{\vec{k}} d_\nu(\vec{k},\vec{q},t) + \frac{\imath}{\hbar\sqrt{(2\pi)^3}} \int d^3r' \int d^3r''\, e^{-\imath\vec{q}\cdot\vec{r}''} f_{ij}(\omega_{\vec{k}}, \vec{r}', \vec{r}'') E^i(\vec{r}',t)\, v_\nu^j(\vec{q}), \tag{21}$$

$$\dot{b}_\nu(\vec{k},\vec{q},t) = \frac{\imath}{\hbar}[\tilde{H}, b_\nu(\vec{k},\vec{q},t)] = -\imath\omega_{\vec{k}} b_\nu(\vec{k},\vec{q},t) + \frac{\mu_0}{\hbar\sqrt{(2\pi)^3}} \int d^3r' \int d^3r''\, e^{-\imath\vec{q}\cdot\vec{r}''} g_{ij}(\omega_{\vec{k}}, \vec{r}', \vec{r}'') H^i(\vec{r}',t) s_\nu^j(\vec{q}). \tag{22}$$

These equations can be solved formally as

$$d_\nu(\vec{k},\vec{q},t) = d_\nu(\vec{k},\vec{q},0)\, e^{-\imath\omega_{\vec{k}}t} + \frac{\imath}{\hbar\sqrt{(2\pi)^3}} \int_0^t dt'\, e^{-\imath\omega_{\vec{k}}(t-t')} \times \int d^3r' \int d^3r''\, e^{-i\vec{q}\cdot\vec{r}''} f_{ij}(\omega_{\vec{k}}, \vec{r}', \vec{r}'') E^i(\vec{r}',t') v_\nu^j(\vec{q}), \tag{23}$$

$$b_\nu(\vec{k},\vec{q},t) = b_\nu(\vec{k},\vec{q},0)\,\mathrm{e}^{-\imath\omega_{\vec{k}}t} + \frac{\mu_0}{\hbar\sqrt{(2\pi)^3}}\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\imath\omega_{\vec{k}}(t-t')}$$
$$\times \int \mathrm{d}^3r' \int \mathrm{d}^3r''\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}''} g_{ij}(\omega_{\vec{k}},\vec{r}',\vec{r}'')H^i(\vec{r}',t')s_\nu^j(\vec{q}). \tag{24}$$

Substituting (23) into (8) and (24) into (9) and using the completeness relations

$$\sum_{\nu=1}^{3} e^i_{\nu\vec{q}}e^j_{\nu\vec{q}} = \sum_{\nu=1}^{3} s^i_{\nu\vec{q}}s^j_{\nu\vec{q}} = \delta_{ij} \tag{25}$$

give us the constitutive equations of the magnetodielectric medium which relate the electric and magnetic polarization densities of the medium to the macroscopic electric and magnetic fields respectively as

$$P_i(\vec{r},t) = P_{Ni}(\vec{r},t) + \varepsilon_0\int_0^{|t|}\mathrm{d}t'\int \mathrm{d}^3r'\,\chi^e_{ij}(\vec{r},\vec{r}',|t|-t')E^j(\vec{r}',\pm t'), \tag{26}$$

$$M_i(\vec{r},t) = M_{Ni}(\vec{r},t) + \int_0^{|t|}\mathrm{d}t'\int \mathrm{d}^3r'\,\chi^m_{ij}(\vec{r},\vec{r}',|t|-t')H^j(\vec{r}',\pm t'), \tag{27}$$

where the upper (lower) sign corresponds to $t > 0$ $(t < 0)$ and $\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \frac{\vec{P}^{\parallel}}{\varepsilon_0}$ is the total electric field. The memory tensors

$$\chi^e(\vec{r},\vec{r}',t) = \begin{cases} \dfrac{8\pi}{\hbar c^3\varepsilon_0}\displaystyle\int_0^\infty \mathrm{d}\omega\,\omega^2\sin\omega t\int \mathrm{d}^3r''[f(\omega,\vec{r},\vec{r}'')\cdot f^t(\omega,\vec{r}',\vec{r}'')] & t > 0\\ 0 & t \leqslant 0 \end{cases} \tag{28}$$

and

$$\chi^m(\vec{r},\vec{r}',t) = \begin{cases} \dfrac{8\pi\mu_0}{\hbar c^3}\displaystyle\int_0^\infty \mathrm{d}\omega\,\omega^2\sin\omega t\int \mathrm{d}^3r''[g(\omega,\vec{r},\vec{r}'')\cdot g^t(\omega,\vec{r}',\vec{r}'')] & t > 0\\ 0 & t \leqslant 0 \end{cases} \tag{29}$$

are respectively the electric and magnetic susceptibility tensors of the magnetodielectric medium which have been obtained in terms of the coupling tensors f, g and their transpositions f^t, g^t. For a medium with a definite pair of tensors χ^e and χ^m it is possible to solve equations (28) and (29) in terms of the coupling tensors f and g using a type of eigenvalue problem [15]. The operators $\vec{P}_N$ and $\vec{M}_N$ in (26) and (27) are the noise electric and magnetic polarization densities and their explicit forms are given by

$$P_{Ni}(\vec{r},t) = \sum_{\nu=1}^{3}\int\frac{\mathrm{d}^3\vec{q}}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3\vec{k}\int \mathrm{d}^3r'\,f_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')$$
$$\times [d_\nu(\vec{k},\vec{q},0)\,\mathrm{e}^{-\imath\omega_{\vec{k}}t+\imath\vec{q}\cdot\vec{r}'} + \mathrm{h.c.}]v^j_\nu(\vec{q}), \tag{30}$$

$$M_{Ni}(\vec{r},t) = i\sum_{\nu=1}^{3}\int\frac{\mathrm{d}^3\vec{q}}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3\vec{k}\int \mathrm{d}^3r'\,g_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')$$
$$\times [b_\nu(\vec{k},\vec{q},0)\,\mathrm{e}^{-\imath\omega_{\vec{k}}t+\imath\vec{q}\cdot\vec{r}'} - \mathrm{h.c.}]s^j_\nu(\vec{q}). \tag{31}$$

From (28) and (29) it is clear that for a given pair of the susceptibility tensors χ^e and χ^m, the solutions of the relations (28) and (29) for the coupling tensors f and g are not unique. In fact for a given pair of χ^e and χ^m, if the tensors f and g satisfy equations (28) and (29), then the coupling tensors

$$\begin{aligned} f'(\omega,\vec{r},\vec{r'}) &= \int \mathrm{d}^3s\, f(\omega,\vec{r},\vec{s})\cdot A^t(\omega,\vec{s},\vec{r'}) \\ g'(\omega,\vec{r},\vec{r'}) &= \int \mathrm{d}^3s\, g(\omega,\vec{r},\vec{s})\cdot A^t(\omega,\vec{s},\vec{r'}) \end{aligned} \tag{32}$$

for any tensor A where satisfy the orthogonality relation

$$\int \mathrm{d}^3r''A(\omega,\vec{s},\vec{r''})\cdot A^t(\omega,\vec{s'},\vec{r''}) = I\delta(\vec{s}-\vec{s'}) \tag{33}$$

also satisfy equations (28) and (29). Although this affects the spacetime dependence of the noise polarizations and therefore the spacetime dependence of the electromagnetic field operators but all of these are equivalent. This means that various choices of the coupling tensors f and g satisfying (28) and (29), for a given pair of the susceptibility tensors χ^e and χ^m, do not affect the commutation relations between the electromagnetic field operators and hence the physical observables. This subject becomes more clear if we compute the commutation relations between the components of the temporal Fourier transformations of the noise polarizations $\vec{P}_N$ and $\vec{M}_N$ using commutation relations (7) and obtain

$$\begin{aligned} \left[\hat{P}_{Ni}(\vec{r},\omega),\hat{P}^{\dagger}_{Nj}(\vec{r'},\omega')\right] &= \frac{\hbar\varepsilon_0}{\pi}Im\left[\hat{\chi}^e_{ij}(\vec{r},\vec{r'},\omega)\right]\delta(\omega-\omega') \\ \left[\hat{M}_{Ni}(\vec{r},\omega),\hat{M}^{\dagger}_{Nj}(\vec{r'},\omega')\right] &= \frac{\hbar}{\mu_0\pi}Im\left[\hat{\chi}^m_{ij}(\vec{r},\vec{r'},\omega)\right]\delta(\omega-\omega'), \end{aligned} \tag{34}$$

where $\hat{\chi}^e$ and $\hat{\chi}^m$ are respectively the temporal fourier transformations of the tensors χ^e and χ^m. The commutation relations (34) are the generalization of those in [8] for an anisotropic magnetodielectric medium with spatial–temporal dispersion. For a given pair of the tensors χ^e and χ^m, various choices of the coupling tensors f and g satisfying the relations (28) and (29) do not affect the commutation relations (34) and therefore the commutation relations between the electromagnetic field operators. Therefore all of the field operators which are obtained with a definite pair of the susceptibility tensors χ^e and χ^m but with different coupling tensors satisfying (28) and (29) are equivalent.

It is clear from (30) and (31) that the explicit forms of the noise polarization densities are known. Also, because the coupling functions f, g are common factors in the noise densities $\vec{P}_N$, $\vec{M}_N$ and the susceptibility tensors χ^e, χ^m, it is clear that the strength of the noise fields is dependent on the strength of χ^e and χ^m which describe the dissipative character of the magnetodielectric medium. When the medium tends to a nonabsorbing one the noise polarization tends to zero and this quantization method is reduced to the quantization in the presence of a nonabsorbing medium [16].

It should be noted that the time derivatives of the polarization fields $\frac{\partial\vec{P}}{\partial t}$ and $\frac{\partial\vec{M}}{\partial t}$ are continuous at time $t=0$ although the absolute value $|t|$ appears in the constitutive equations (26) and (27). In fact another solution for the Heisenberg equation (21) can be written as

$$\begin{aligned} d_\nu(\vec{k},\vec{q},t) = d^{in}_\nu(\vec{k},\vec{q})\,\mathrm{e}^{-\imath\omega_{\vec{k}}t} + \frac{\imath}{\hbar\sqrt{(2\pi)^3}}\int_{-\infty}^{t}\mathrm{d}t'\,\mathrm{e}^{-\imath\omega_{\vec{k}}(t-t')} \\ \times \int \mathrm{d}^3r'\int \mathrm{d}^3r''\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r''}} f_{ij}(\omega_{\vec{k}},\vec{r'},\vec{r''})E^i(\vec{r'},t')v^j_\nu(\vec{q}), \end{aligned} \tag{35}$$

where $d^{in}_\nu(\vec{k},\vec{q})$ are some time-independent annihilation operators which satisfy the same commutation relations (7). If we find $d_\nu(\vec{k},\vec{q},0)$ from equation (35) and substitute it into the noise polarization field $\vec{P}_N(\vec{r},t)$ given by (30), we deduce

$$P_{Ni}(\vec{r},t) = P^{in}_{Ni}(\vec{r},t) + \varepsilon_0 \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3r' \chi^e_{ij}(\vec{r},\vec{r'},t-t')E^j(\vec{r'},t') \\ - \varepsilon_0 \int_0^{|t|} \mathrm{d}t' \int \mathrm{d}^3r' \chi^e_{ij}(\vec{r},\vec{r'},|t|-t')E^j(\vec{r'},\pm t'), \tag{36}$$

where the susceptibility tensor χ^e is given by (28) and $\vec{P}^{in}_N$ is the same as $\vec{P}_N$ with the exception that the annihilation operators $d_\nu(\vec{k},\vec{q},0)$ should be replaced by $d^{in}_\nu(\vec{k},\vec{q})$. Now substituting $\vec{P}_N$ from (36) into (26) gives us the expression

$$P_i(\vec{r},t) = P^{in}_{Ni}(\vec{r},t) + \varepsilon_0 \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3r' \chi^e_{ij}(\vec{r},\vec{r'},t-t')E^j(\vec{r'},t') \tag{37}$$

for the polarization field $\vec{P}$ which is valid for both positive and negative times. From (37) it is clear that $\frac{\partial\vec{P}}{\partial t}$ and accordingly the electromagnetic field operators are continuous at $t = 0$. One can apply the constitutive equation (37) and a similar equation for the magnetic polarization $\vec{M}$ and use the temporal Fourier transformation, or apply the constitutive equations (26) and (27) and use the Laplace transformation, to solve the coupled constitutive and Maxwell equations. Here we prefer the later, since in this way it is easier to show the limiting cases in the absence of any medium or in the presence of a nonabsorbing medium [16, 17]. Using the forward and backward Laplace transformations and applying the constitutive equations (26) and (27), we can obtain the explicit forms of the electromagnetic operators for both negative and positive times which are continuous at $t = 0$ [12].

4. The solution of the Heisenberg equation

In this section we solve the coupled Maxwell and constitutive equations (18)–(20), (26) and (27) for a translationally invariant medium, that is for a medium that its electric and magnetic susceptibility tensors are dependent on the difference $\vec{r}-\vec{r'}$. For such a medium we can easily obtain the explicit forms of the electromagnetic field operators for both positive and negative times by the spatial Fourier transformation and the temporal Laplace transformation. In this section we solve the coupled Maxwell and constitutive equations for positive times using the forward Laplace transformation. The solution of the Heisenberg equations for negative times can be found similarly from the backward Laplace transformation [12]. Let us define $\underline{\hat{\vec{F}}}$ for any vector field $F(\vec{r},t)$ by

$$\underline{\hat{\vec{F}}}(\vec{k},\rho) = \int \mathrm{d}^3r \int_0^\infty \mathrm{d}t\, \vec{F}(\vec{r},t)\, \mathrm{e}^{-\imath\vec{k}\cdot\vec{r}-\rho t}. \tag{38}$$

Now applying such a transformation on both the sides of equations (18), (26) and (27), together with the Maxwell equations (19) and (20), and then their combination for a translationally invariant medium we find

$$-\imath\vec{k}\times\underline{\hat{\vec{E}}} = -\rho\underline{\hat{\mu}}(\vec{k},p)\underline{\hat{\vec{H}}} - \mu_0\rho\underline{\hat{\vec{M}}}_N(\vec{k},\rho) + \underline{\vec{B}}(\vec{k},0), \\ -\imath\vec{k}\times\underline{\hat{\vec{H}}} = \rho\underline{\hat{\varepsilon}}(\vec{k},\rho)\underline{\hat{\vec{E}}} + \rho\underline{\hat{\vec{P}}}_N(\vec{k},\rho) - \underline{\vec{D}}(\vec{k},0), \tag{39}$$

where $\underline{\vec{B}}(\vec{k},0)$, $\underline{\vec{D}}(\vec{k},0)$ are respectively the spatial Fourier transformations of $\vec{B}(\vec{r},0)$ and $\vec{D}(\vec{r},0)$ and $\underline{\hat{\varepsilon}} = \varepsilon_0(1+\underline{\hat{\chi}}^e)$, $\underline{\hat{\mu}} = \mu_0(1+\underline{\hat{\chi}}^m)$ are the transformations introduced in (38) for the permittivity and permeability tensors of the medium. Equations (39) can be written in a compact form by using a matrix notation as follows:

$$\Lambda(\vec{k},\rho)\begin{bmatrix}\underline{\hat{\tilde{E}}}\\ \underline{\hat{\tilde{H}}}\end{bmatrix}=\begin{bmatrix}\mu_0\rho\underline{\hat{\tilde{M}}}_N-\underline{\vec{B}}(\vec{k},0)\\ -\rho\underline{\hat{\tilde{P}}}_N+\underline{\vec{D}}(\vec{k},0)\end{bmatrix}, \tag{40}$$

where $\Lambda(\vec{k},\rho)$ is a 6×6 matrix defined by

$$\Lambda(\vec{k},\rho)=\begin{bmatrix}O(\vec{k}) & -\rho\underline{\hat{\mu}}(\vec{k},\rho)\\ \rho\underline{\hat{\varepsilon}}(\vec{k},\rho) & O(\vec{k})\end{bmatrix}, \tag{41}$$

and

$$O(\vec{k})=\begin{bmatrix}0 & -\imath k_3 & \imath k_2\\ \imath k_3 & 0 & -\imath k_1\\ -\imath k_2 & \imath k_1 & 0\end{bmatrix}. \tag{42}$$

Finally, substituting $\underline{\hat{\tilde{P}}}_N(\vec{k},\rho)$, $\underline{\hat{\tilde{M}}}_N(\vec{k},\rho)$, $\vec{D}(\vec{k},0)$ and $\vec{B}(\vec{k},0)$ into the right-hand side of (40) using the expansions (1), (4), (30) and (31) and multiplying equation (40) on the left-hand side by $\Lambda^{-1}(\vec{k},\rho)$, we find

$$\begin{aligned}E_i(\vec{r},t)={}&\imath\sum_{\lambda=1}^{2}\int \mathrm{d}^3k\sqrt{\frac{\hbar\omega_{\vec{k}}\varepsilon_0}{2(2\pi)^3}}[\gamma_{ij}(\vec{k},t)a_{\vec{k}\lambda}(0)\,\mathrm{e}^{\imath\vec{k}\cdot\vec{r}}-\text{h.c.}]e^j_{\vec{k}\lambda}\\
&+\imath\sum_{\lambda=1}^{2}\int \mathrm{d}^3k\sqrt{\frac{\hbar\omega_{\vec{k}}\mu_0}{2(2\pi)^3}}[\xi_{ij}(\vec{k},t)a_{\vec{k}\lambda}(0)\,\mathrm{e}^{\imath\vec{k}\cdot\vec{r}}-\text{h.c.}]s^j_{\vec{k}\lambda}\\
&+\imath\sum_{\nu=1}^{3}\int \mathrm{d}^3q\int\frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3}}[\zeta_{ij}(\omega_{\vec{q}},\vec{k},t)b_\nu(\vec{q},\vec{k},0)\,\mathrm{e}^{\imath\vec{k}\cdot\vec{r}}-\text{h.c.}]s^j_{\vec{k}\nu}\\
&+\sum_{\nu=1}^{3}\int \mathrm{d}^3q\int\frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3}}[\eta_{ij}(\omega_{\vec{q}},\vec{k},t)d_\nu(\vec{q},\vec{k},0)\,\mathrm{e}^{\imath\vec{k}\cdot\vec{r}}+\text{h.c.}]e^j_{\vec{k}\nu},\end{aligned} \tag{43}$$

where $\omega_{\vec{k}}=c|\vec{k}|$, $\omega_{\vec{q}}=c|\vec{q}|$ and we have used $\Lambda(-\vec{k},\rho)=\Lambda^*(\vec{k},\rho)$. The tensors γ,ξ,ζ and η are given by

$$\begin{aligned}\gamma_{ij}(\vec{k},t)&=L^{-1}\big[[\Lambda(\vec{k},\rho)]^{-1}_{i(j+3)}\big],\\
\xi_{ij}(\vec{k},t)&=-L^{-1}\big[[\Lambda(\vec{k},\rho)]^{-1}_{ij}\big],\\
\zeta_{ij}(\omega_{\vec{q}},\vec{k},t)&=\mu_0L^{-1}\left[\frac{\rho}{\rho+\imath\omega_{\vec{q}}}\sum_{l=1}^{3}\big([\Lambda(\vec{k},\rho)]^{-1}_{il}\,\underline{g}_{lj}(\omega_{\vec{q}},\vec{k})\big)\right],\\
\eta_{ij}(\omega_{\vec{q}},\vec{k},t)&=-L^{-1}\left[\frac{\rho}{\rho+\imath\omega_{\vec{q}}}\sum_{l=1}^{3}\big([\Lambda(\vec{k},\rho)]^{-1}_{i(l+3)}\,\underline{f}_{lj}(\omega_{\vec{q}},\vec{k})\big)\right],\end{aligned} \tag{44}$$

where $L^{-1}\{f(\rho)\}$ denotes the inverse Laplace transform of $f(\rho)$ and the tensors $\underline{f},\underline{g}$ are given by

$$\underline{f}_{ij}(\omega,\vec{k})=\int \mathrm{d}^3r f_{ij}(\omega,\vec{r})\,\mathrm{e}^{-\imath\vec{k}\cdot\vec{r}}\qquad \underline{g}_{ij}(\omega,\vec{k})=\int \mathrm{d}^3r g_{ij}(\omega,\vec{r})\,\mathrm{e}^{-\imath\vec{k}\cdot\vec{r}}. \tag{45}$$

Similarly for the magnetic field $\vec{H}$ we obtain

$$H_i(\vec{r},t) = \imath\sum_{\lambda=1}^{2}\int d^3k\sqrt{\frac{\hbar\omega_{\vec{k}}\varepsilon_0}{2(2\pi)^3}}[\tilde{\gamma}_{ij}(\vec{k},t)a_{\vec{k}\lambda}(0)\,e^{\imath\vec{k}\cdot\vec{r}} - \text{h.c.}]e^j_{\vec{k}\lambda}$$
$$+\imath\sum_{\lambda=1}^{2}\int d^3k\sqrt{\frac{\hbar\omega_{\vec{k}}\mu_0}{2(2\pi)^3}}[\tilde{\xi}_{ij}(\vec{k},t)a_{\vec{k}\lambda}(0)\,e^{\imath\vec{k}\cdot\vec{r}} - \text{h.c.}]s^j_{\vec{k}\lambda}$$
$$+\imath\sum_{\nu=1}^{3}\int d^3q\int\frac{d^3k}{\sqrt{(2\pi)^3}}[\tilde{\zeta}_{ij}(\omega_{\vec{q}},\vec{k},t)b_\nu(\vec{q},\vec{k},0)\,e^{\imath\vec{k}\cdot\vec{r}} - \text{h.c.}]s^j_{\vec{k}\nu}$$
$$+\sum_{\nu=1}^{3}\int d^3q\int\frac{d^3k}{\sqrt{(2\pi)^3}}[\tilde{\eta}_{ij}(\omega_{\vec{q}},\vec{k},t)d_\nu(\vec{q},\vec{k},0)\,e^{\imath\vec{k}\cdot\vec{r}} + \text{h.c.}]e^j_{\vec{k}\nu}, \tag{46}$$

where

$$\tilde{\gamma}_{ij}(\vec{k},t) = L^{-1}\big[[\Lambda(\vec{k},\rho)]^{-1}_{(i+3)(j+3)}\big],$$
$$\tilde{\xi}_{ij}(\vec{k},t) = -L^{-1}\big[[\Lambda(\vec{k},\rho)]^{-1}_{(i+3)j}\big],$$
$$\tilde{\zeta}_{ij}(\omega_{\vec{q}},\vec{k},t) = \mu_0 L^{-1}\left[\frac{\rho}{\rho+\imath\omega_{\vec{q}}}\sum_{l=1}^{3}\big([\Lambda(\vec{k},\rho)]^{-1}_{(i+3)l}\underline{g}_{lj}(\omega_{\vec{q}},\vec{k})\big)\right], \tag{47}$$
$$\tilde{\eta}_{ij}(\omega_{\vec{q}},\vec{k},t) = -L^{-1}\left[\frac{\rho}{\rho+\imath\omega_{\vec{q}}}\sum_{l=1}^{3}\big([\Lambda(\vec{k},\rho)]^{-1}_{(i+3)(l+3)}\underline{f}_{lj}(\omega_{\vec{q}},\vec{k})\big)\right].$$

Expressions (43)–(47) and the commutation relations (3) and (7) show that the commutation relations between components of the operators $\vec{E}$ and $\vec{H}$ are independent of the various choices of the coupling tensors f and g satisfying the relations (28) and (29) for a given pair of the tensors χ^e and χ^m. Finally having the fields $\vec{E}$ and $\vec{H}$ we can obtain the polarization fields $\vec{P}$ and $\vec{M}$ from the constitutive equations (26) and (27).

It should be pointed out that hereunto we have assumed that the medium is a polarizable and magnetizable insulator one. In this case the operator $\vec{P}$ defined by (8) is the electric polarization density and $\frac{\partial\vec{P}}{\partial t}$ is the current density induced in the medium due to the electric polarization. When we are concerned with a conductor magnetodielectric medium the fields $\vec{P}$ and $\vec{D}$ in (18) may not be interpreted as the electric polarization and displacement fields and the quantity $\frac{\partial\vec{P}}{\partial t}$ in the Maxwell equation (19), which can be rewritten as

$$\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t} = \nabla\times\vec{H}, \tag{48}$$

is not merely the current source created by the electric polarization but is the sum of the free current density, due to the motion of the free charges, and the current due to the polarizability of the medium. In this case if we compute $\dot{d}_\nu(\vec{k},\vec{q},t)$ from (23) and substitute it into (8), instead of the constitutive equation (26), we obtain the linear responsive relation

$$\frac{\partial P_i(\vec{r},t)}{\partial t} = J_{Ni}(\vec{r},t) \pm \int_0^{|t|} dt'\int d^3r'\,Q_{ij}(\vec{r},\vec{r'},|t|-t')E^j(\vec{r'},\pm t'), \tag{49}$$

where now

$$Q(\vec{r},\vec{r}',t)=\begin{cases}\dfrac{8\pi}{\hbar c^3}\displaystyle\int_0^\infty \mathrm{d}\omega\omega^3\cos\omega t\int \mathrm{d}^3r''[f(\omega,\vec{r},\vec{r}'')\cdot f^t(\omega,\vec{r}',\vec{r}'')] & t>0\\ 0 & t\leqslant 0\end{cases} \tag{50}$$

is $\varepsilon_0\frac{\partial\chi^e(\vec{r},\vec{r}',t)}{\partial t}+\sigma(\vec{r},\vec{r}',t)$, with χ^e and σ respectively the electric susceptibility and conductivity tensors of the medium and

$$J_{Ni}(\vec{r},t)=\frac{\partial P_{Ni}}{\partial t}=-\imath\sum_{\nu=1}^{3}\int\frac{\mathrm{d}^3\vec{q}}{\sqrt{(2\pi)^3}}\int \mathrm{d}^3\vec{k}\int \mathrm{d}^3r'\omega_{\vec{k}}f_{ij}(\omega_{\vec{k}},\vec{r},\vec{r}')$$
$$\times[d_\nu(\vec{k},\vec{q},0)\,\mathrm{e}^{-\imath\omega_{\vec{k}}t+\imath\vec{q}.\vec{r}'}-\mathrm{h.c.}]v_\nu^j(\vec{q}) \tag{51}$$

is a noise current density. In this case the explicit forms of the electromagnetic field are clearly the same as the relations (43)–(47) with the exception that we should replace $\rho\underline{\hat{\varepsilon}}(\vec{k},\rho)$ by $\rho\underline{\hat{\varepsilon}}(\vec{k},\rho)+\underline{\hat{\sigma}}(\vec{k},\rho)$ where $\underline{\hat{\sigma}}$ is the transformation introduced in (38) for the conductivity tensor σ.

5. Summary

By modeling an anisotropic and inhomogeneous magnetodielectric medium with two independent quantum fields, namely E and M quantum fields, we could investigate electromagnetic field quantization in the presence of such a medium consistently. The electric and magnetic susceptibility tensors χ^e and χ^m of the medium were introduced in terms of the coupling tensors which couple the electromagnetic field to E and M quantum fields respectively. The explicit spacetime dependence of the noise polarizations was obtained in terms of the ladder operators of the medium and the coupling tensors as a consequence of Heisenberg equations. In this approach, both the Maxwell and constitutive equations were obtained as Heisenberg equations of the total system. In the limiting case, i.e., when there is no medium, this approach tends to the usual quantization of the electromagnetic field in free space. Also when the medium approaches a nonabsorptive one, the noise polarizations tend to zero and this quantization scheme is reduced to the known quantization in such a medium.

References

[1] Grunner T and Welsch D G 1995 *Phys. Rev.* A **51** 3246
[2] Matloob R, Loudon R and Barnett S M 1995 *Phys. Rev.* A **52** 4823
[3] Grunner T and Welsch D G 1996 *Phys. Rev.* A **53** 1818
[4] Grunner T and Welsch D G 1996 *Phys. Rev.* A **54** 1661
[5] Matloob R and Loudon R 1996 *Phys. Rev.* A **53** 4567
[6] Dung H T, Knoll L and Welsch D G 1998 *Phys. Rev.* A **57** 3931
[7] Scheel S, Knoll L and Welsch D G 1998 *Phys. Rev.* A **58** 700
[8] Dung H T, Buhmann S Y, Knö L and Welsh D G 2003 *Phys. Rev.* A **68** 043816
[9] Hopfield J J 1958 *Phys. Rev.* **112** 1555
[10] Fano U 1956 *Phys. Rev.* **103** 1202
[11] Huttner B and Barnett S M 1992 *Phys. Rev.* A **46** 4306
[12] Suttorp L G and Wubs M 2004 *Phys. Rev.* A **70** 013816
[13] Suttorp L G 2007 *J. Phys. A. Math. Theor.* **40** 13
[14] Landau L D and Lifshitz E M 1977 *Electrodynamics of Continuous Media* (Oxford: Pergamon)
[15] Raabe C, Scheel S and Welsch D G 2007 *Phys. Rev.* A **75** 053813
[16] Kheirandish F and Amooshahi M 2006 *Phys. Rev.* A **74** 042102
[17] Amooshahi M and Kheirandish F 2007 *Phys. Rev.* A **76** 062103
[18] Schleich W P 2001 *Quantum Optics in Phase Space* (New York: Wiley-VCH)

[19] Milonni P W 1994 *The Quantum Vacuum* (New York: Academic)
[20] Ford G W, Lewis J T and O'Connell R F 1988 *Phys. Rev.* A **37** 4419
[21] Ford G W, Lewis J T and O'Connell R F 1985 *Phys. Rev. Lett.* **55** 2273
[22] Caldeira A O and Leggett A J 1981 *Phys. Rev. Lett.* **46** 211
[23] Caldeira A O and Leggett A J 1983 *Ann. Phys., NY* **149** 374
[24] Castro Neto A H and Caldeira A o 1991 *Phys. Rev. Lett.* **67** 1960
[25] Caldeira A O and Leggett A J 1984 *Ann. Phys., NY* **153** 445
[26] Shuster H G and Veria V R 1986 *Phys. Rev.* B **34** 189
[27] Ford G W, Kac M and Mazur P 1964 *J. Math. Phys.* **6** 504
[28] Ford G W, Lewis J T and O'Connel R F 1987 *Phys. Rev.* A **37** 4419
Ford G W, Lewis J T and O'Connel R F 2007 *Phys. Rev.* A **37** 053813